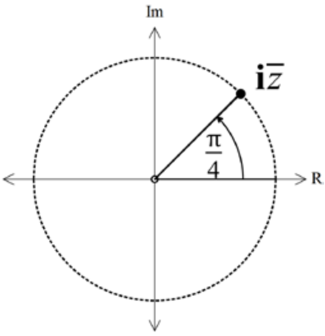
Section 1: Multiple Choice (10 marks)
Indicate your answers on the answer sheet provided.

Q1. Given the complex number z has $Arg\, z = \dfrac{\pi}{4}$, which of the following is a correct representation of $i\,\bar{z}$?

A



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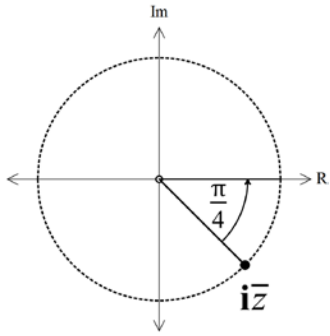

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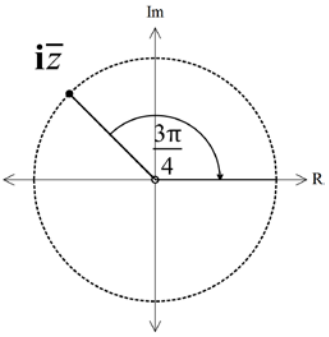
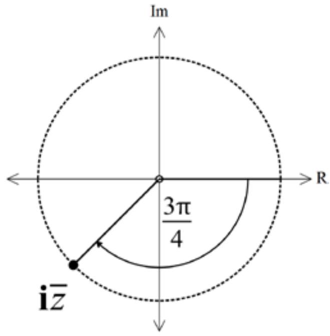


D



Q2. Which expression is equal to $\int_0^a [f(a-x) + f(a+x)]dx$?

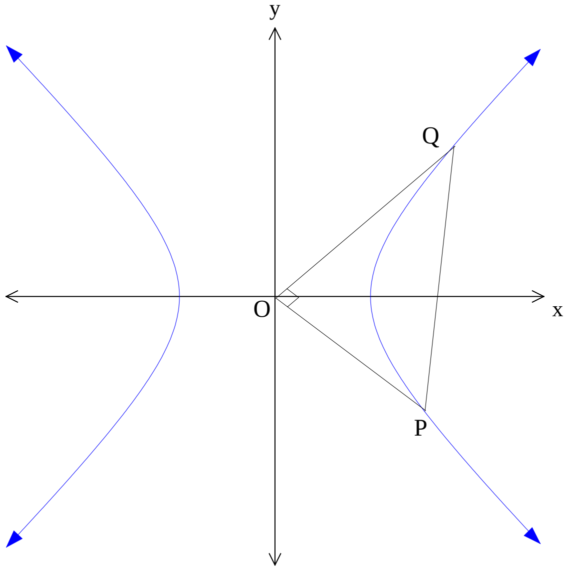
A $\int_0^a f(x)dx$

B $\int_0^{2a} f(x)dx$

C $2\int_0^a f(x)dx$

D $\int_{-a}^a f(x)dx$

Q3. The diagram below shows the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $a > b > 0$. The points $(a\sec\theta, b\tan\theta)$ and $Q\ (a\sec\alpha, b\tan\alpha)$ lie on the hyperbola and the chord PQ subtends a right angle at the origin.



Use the parametric representation of the hyperbola to determine which of the following expressions is correct?

A $\quad \sin\theta\sin\alpha = -\frac{a^2}{b^2}$

B $\quad \sin\theta\sin\alpha = \frac{a^2}{b^2}$

C $\quad \tan\theta\tan\alpha = -\frac{a^2}{b^2}$

D $\quad \tan\theta\tan\alpha = \frac{a^2}{b^2}$

Q4. The polynomial equation $x^3 + 3x^2 - 2x + 6 = 0$ has roots α, β and γ.

Which of the following polynomials has roots $\alpha - 1, \beta - 1, \gamma - 1$?

A $\quad x^3 - 5x + 10 = 0$

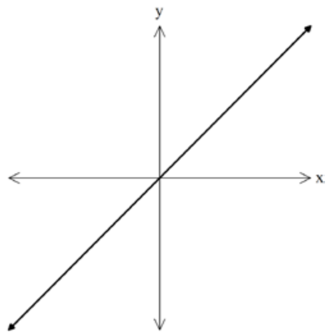
B $\quad x^3 + 3x^2 - 2x + 14 = 0$

C $\quad x^3 + 6x^2 + x + 8 = 0$

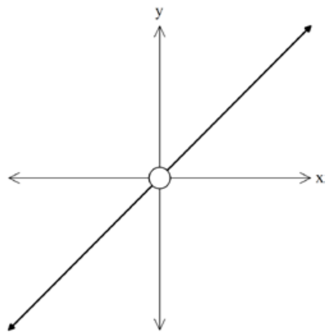
D $\quad x^3 + 6x^2 + 7x + 8 = 0$

Q5. Which of the following graphs best represents the implicit function $\frac{x}{y} + \frac{y}{x} = 2$?

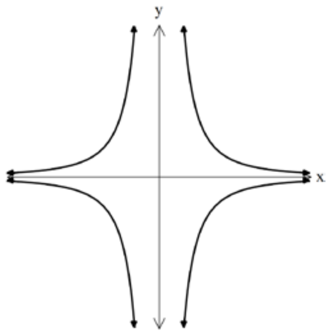
A



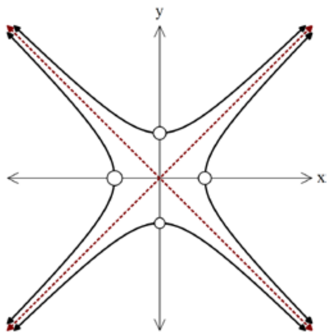
B



C



D



Q6. Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{cos\theta + 2sin\theta + 3}\, d\theta$ using the substitution $t = tan\frac{\theta}{2}$

A 0.322

B 0.785

C 1.107

D 1.570

Q7. The point $P(z)$ moves on the complex plane according to the condition $|z-i|+|z+i|=4$. The Cartesian equation of the locus of P is:

A $\quad \frac{x^2}{4}+\frac{y^2}{3}=1$

B $\quad \frac{x^2}{3}+\frac{y^2}{4}=1$

C $\quad x^2+y^2=1$

D $\quad x^2+y^2=4$

Q8. The region bounded by $y \le 4x^2 - x^4$ and $0 \le x \le 2$ is rotated about the y axis to form a solid. What is the volume of this solid using the method of cylindrical shells?

A $\quad \frac{16\pi}{3}$ units^3

B $\quad \frac{8\pi}{3}$ units^3

C $\quad \frac{20\pi}{3}$ units^3

D $\quad \frac{32\pi}{3}$ units^3

Q9. What is the area of the largest rectangle that can be inscribed in the ellipse $4x^2+9y^2=36$?

A $\quad 24\sqrt{2}$

B $\quad 6\sqrt{2}$

C $\quad 24$

D $\quad 12$

Q10. Consider the equation $\left(\frac{2+i}{c}\right)^p=1$ where c is a real and $p \neq 0$

For how many values of c will this equation have real solutions?

A None

B One

C Two

D Four

End of Multiple Choice

Section II (90 marks)

Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.
In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 Use a SEPARATE writing booklet. (15 marks)

(a) Find $\int tan^4 x sec^2 x dx$ *(1)*

(b) Use the substitution $u = x - 2$ to evaluate $\displaystyle\int_{\frac{3}{2}}^{\frac{5}{2}} \frac{1}{\sqrt{(x-1)(3-x)}}\,dx$ *(3)*

(c) (i) Write $\dfrac{3x+2}{x^2+5x+6}$ as a sum of partial fractions. *(2)*

(ii) Hence evaluate $\displaystyle\int_0^2 \frac{3x+2}{x^2+5x+6}\,dx$ *(2)*

(d) Given that the point P$(6cos\theta, 2sin\theta)$ lies on an ellipse, determine the following:

(i) The eccentricity. *(1)*

(ii) Coordinates of the foci. *(1)*

(iii) Equations to the directrices. *(1)*

(iv) Sketch neatly this ellipse showing all important features. *(1)*

(e) For the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{25} = 1$

(i) Find the eccentricity. *(1)*

(ii) Find the equations of the asymptotes. *(1)*

(iii) If P is on the hyperbola and S and S’ are its foci, then given PS=2, find PS’. *(1)*

Question 12 Use a SEPARATE writing booklet. (15 marks)

(a) Realise the denominator of $\dfrac{7 - 2\mathbf{i}}{3 + \mathbf{i}}$ *(1)*

(b) (i) On the same diagram, sketch the locus of both $|z - 2| = 2$ and $|z| = |z - 4i|$. *(2)*

(ii) What is the complex number represented by the point of intersection of these two loci? *(1)*

(c) Let z be a complex number of modulus 3 and ω be a complex number of modulus 1.

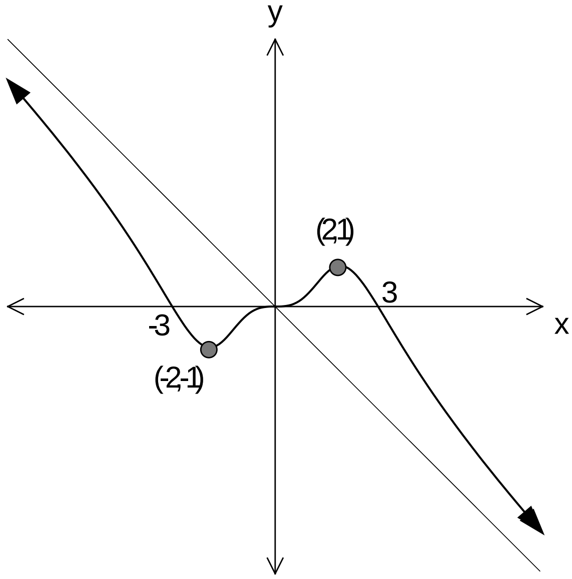
Show that $|z - \omega|^2 = 10 - \left(z\overline{\omega} + \overline{z}\omega\right)$ *(2)*

(d) Given the polynomial $2x^3 + 3x^2 - x + 1 = 0$ has roots α, β and γ:

(i) Find the polynomial whose roots are α^2, β^2 and γ^2. *(2)*

(ii) Determine the value of $\alpha^3 + \beta^3 + \gamma^3$. *(2)*

(e) The graph of $y = f(x)$ is shown. The line $y = -x$ is an oblique asymptote to the curve.



Use separate one-third page graphs, to sketch:

(i) $f(-x)$ *(1)*

(ii) $f(|x|)$ *(1)*

(iii) $\dfrac{x}{f(x)}$ *(3)*

Question 13 Use a SEPARATE writing booklet. (15 marks)

(a) Given that $1 + i$ is a zero of P(x) = 0 where $P(x) = x^4 - x^3 - 2x^2 + 6x - 4$, factorise $P(x)$ fully over the field of the complex numbers. *(2)*

(b) The ellipse $\frac{(x-4)^2}{9} + \frac{y^2}{4} = 1$ is rotated about the y axis to form a solid of revolution.

(i) By taking slices perpendicular to the axis of rotation, show that the volume of a slice is $8\pi\sqrt{36 - 9y^2}\,\delta y$ *(2)*

(ii) Find the exact volume of the solid. *(2)*

(c) Find all the roots of the equation $18x^3 + 3x^2 - 28x + 12 = 0$, given that two of the roots are equal. *(3)*

(d) P and Q are two points on the same branch of the rectangular hyperbola $xy = 25$. Given that P is $(5p, \frac{5}{p})$ and Q is $(5q, \frac{5}{q})$:

(i) Show that PQ has the equation $x + pqy = 5(p+q)$ where P and Q are parameters p and q respectively. *(2)*

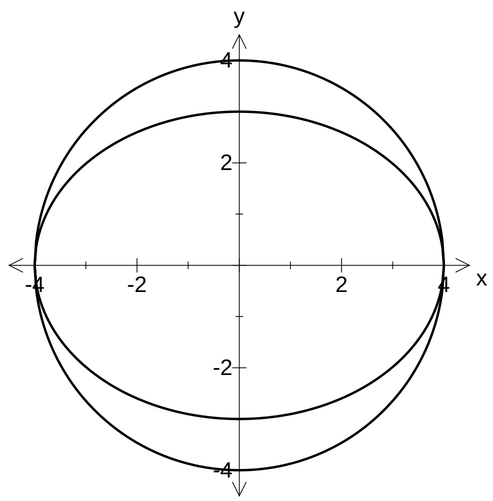
(ii) If PQ has a constant length of m^2, show that

$$25[(p+q)^2 - 4pq]\,(p^2q^2 + 1) = m^4p^2q^2$$ *(2)*

(iii) Show that the locus of R, the midpoint of PQ, in Cartesian form is $xym^4 = 4(xy - 25)(x^2 + y^2)$. *(2)*

Question 14 Use a SEPARATE writing booklet. (15 marks)

(a) The diagram shows an ellipse with equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and the larger of its auxiliary circles. The coordinates of a point P on the ellipse are ($4\cos\theta$, $3\sin\theta$) where $\theta \neq 0$ or π.



A straight line l parallel to the y axis intersects the x axis at N and the ellipse and the auxiliary circle at the points P and Q respectively.

(i) Find the equations of the tangent to the ellipse at P and to the auxiliary circle at Q. *(4)*

(ii) The tangents at P and Q intersect at point R. Show that R lies on the x axis. *(2)*

(iii) Prove that ON × OR is independent of the positions of P and Q. *(1)*

(b) By expanding $(\cos\theta + i\sin\theta)^3$ it can be shown that $\cot 3\theta = \frac{t^3 - 3t}{3t^2 - 1}$ where $t = \cot\theta$. (Do **NOT** prove this.)

(i) Solve $\cot 3\theta = -1$ for $0 \leq \theta \leq 2\pi$ *(2)*

(ii) Hence show that $\cot\frac{\pi}{12} . \cot\frac{5\pi}{12} . \cot\frac{9\pi}{12} = -1$ *(2)*

(iii) Write down a cubic equation with roots $\tan\frac{\pi}{12}$, $\tan\frac{5\pi}{12}$ and $\tan\frac{9\pi}{12}$ *(1)*
(Express your answer as a polynomial equation with positive integer coefficients).

(c) (i) Show that the derivative of the function $y = x^x$ for $x > 0$ is $(\ln x + 1)\, x^x$. *(1)*

(ii) Hence or otherwise neatly draw $y = x^x$ for $x > 0$ *(2)*

Question 15 Use a SEPARATE writing booklet. (15 marks)

(a) A triangle ABC is right-angled at A and it has sides of lengths a, b and c units. A circle of radius r units is drawn so that the sides of the triangle are tangents to the inscribed circle.

Prove that $r = \frac{1}{2}(c + b - a)$. *(3)*

(b) (i) A solid has as its base the region bounded by the curves $y = x$ and $x = 2y - \frac{y^2}{2}$.

Cross sections parallel to the x axis are equilateral triangles with a side in the base.

Show that the volume is given by $V = \frac{\sqrt{3}}{4}\int_0^2 (y^2 + \frac{y^4}{4} - y^3)dy$ *(3)*

(ii) Calculate the volume of this solid. *(2)*

(c) (i) If $I_n = \int_0^1 (1 - x^2)^{\frac{n}{2}}\, dx$ where n is a positive integer, show that *(3)*

$$I_n = \frac{n}{(n+1)} I_{n-2} .$$

(ii) Hence evaluate I_5. *(2)*

(d) Find the exact range of values for m, where m is a non-negative real number, such that $x^2 + (2 - m^2)y^2 = 1$ represents an ellipse in the Cartesian Plane. *(2)*

PTO for Q16

Question 16 Use a SEPARATE writing booklet. (15 marks)

(a) (i) Show that $a^2 + b^2 \geq 2ab$ where a and b are distinct positive real numbers. *(1)*

(ii) Hence show that $a^2 + b^2 + c^2 \geq ab + ac + bc$. *(1)*

(iii) Hence show that $\sin^2\alpha + \cos^2\alpha \geq \sin 2\alpha$. *(1)*

(iv) Hence show that $\sin^2\alpha + \cos^2\alpha + \tan^2\alpha \geq \sin\alpha - \cos\alpha + \sec\alpha + \frac{1}{2}\sin 2\alpha$. *(1)*

(b) An orchestra has *2n* cellists; *n* being female and *n* male. From the *2n* cellists a committee of three members is formed which contains more females than males. Two members of the orchestra cellists are Paul and Matilda. Find the probability that a committee chosen at random has Matilda in it if it is known that Paul has been chosen. *(1)*

(c) When a polynomial P(x) is divided by $x^2 - a^2$, where $a \neq 0$, the remainder is $px + q$.

(i) Show that $p = \frac{1}{2a}[P(a) - P(-a)]$ and $q = \frac{1}{2}[P(a) + P(-a)]$ *(2)*

(ii) Find the remainder when $P(x) = x^n - a^n$ is divided by $x^2 - a^2$ and n is a positive integer. *(2)*

(d) For n = 1, 2, 3, …… let $S_n = 1 + \sum_1^n \frac{1}{r!}$.

(i) Prove by Mathematical Induction that $e - S_n = e\int_0^1 \frac{x^n}{n!}e^{-x}dx$ *(3)*

(ii) Deduce that $0 < e - S_n < \frac{3}{(n+1)!}$ for n =1, 2, 3, …….. *(1)*

(iii) Prove that $(e - S_n)n!$ is not an integer for n = 2, 3, 4, …….. *(1)*

(iv) Show that there cannot exist positive integers p and q such that $e = \frac{p}{q}$. *(1)*

END OF EXAM

Multiple Choice

1. A
2. B
3. A
4. D
5. B
6. A
7. B
8. D
9. D
10. C

conjugate of z followed by 90° anticlockwise rotation

$\frac{x^2}{9} - \frac{y^2}{4} = 1 \qquad e^2 = \frac{13}{4} \qquad e = \frac{\sqrt{13}}{2}$

(5) $x^2 + y^2 = 2xy \qquad (x-y)^2 = 0 \qquad y = x \qquad x \neq 0, y \neq 0$

$y = x - 1 \qquad \therefore x = y + 1 \qquad (y+1)^3 + 3(y+1)^2 - 2(y+1) + 6 =$

$\delta V = \pi (R^2 - r^2)\,\delta x = \pi [1 - \sin^2 x]\,dx$

$\frac{x^2}{9} + \frac{y^2}{4} = 1 \qquad \text{Area} = (2x)(2y) = 4xy$

$\left| \frac{2+i}{c} \right| = 1$

$|c| = |2+i| = \sqrt{5}$

since $c \in R$, $c = \pm\sqrt{5}$

Question 11

(a) $\int \tan^4 x \sec^2 x\,dx = \frac{1}{5}\tan^5 x + C$ (1)

(b) $u = x - 2$ when $x = 3/2$, $u = -1/2$

$u + 1 = x - 1$ $x = 5/2$, $u = 1/2$

$1 - u = 3 - x$

$$\therefore \int_{3/2}^{5/2} \frac{1}{\sqrt{(x-1)(3-x)}}\,dx \Rightarrow \int_{-1/2}^{1/2} \frac{1}{\sqrt{(1+u)(1-u)}}\,du \quad (1)$$

$$= \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \left[\sin^{-1} u\right]_{-1/2}^{1/2} \quad (1)$$

$$= \sin^{-1}(1/2) - \sin^{-1}(-1/2)$$

$$= \frac{\pi}{6} - -\frac{\pi}{6} = \pi/3 \quad (1)$$

(c) (i) $\dfrac{3x+2}{(x+3)(x+2)} \equiv \dfrac{A}{x+3} + \dfrac{B}{x+2}$

$3x+2 = A(x+2) + B(x+3)$

when $x=-2$; $-4 = B$

when $x=-3$; $-7 = -A$ $\therefore A = 7$ (1)

$\therefore \dfrac{+7}{x+3} - \dfrac{4}{x+2}$ (1)

(ii) $\displaystyle\int_0^2 \frac{3x+2}{(x+3)(x+2)}\,dx = \int_0^2 \left(\frac{7}{x+3} - \frac{4}{x+2}\right) dx$

$= \Big[7\ln(x+3) - 4\ln(x+2)\Big]_0^2$ (1)

$= 7\ln 5 - 4\ln 4 - 7\ln 3 + 4\ln 2$

$= 7\ln(5/3) - 4\ln(2)$ (1)

or// $= \ln\left(\dfrac{78125}{34992}\right)$

d) (i) $P(6\cos\theta, 2\sin\theta)$ $\therefore \dfrac{x^2}{36} + \dfrac{y^2}{4} = 1$

$b^2 = a^2(1-e^2)$

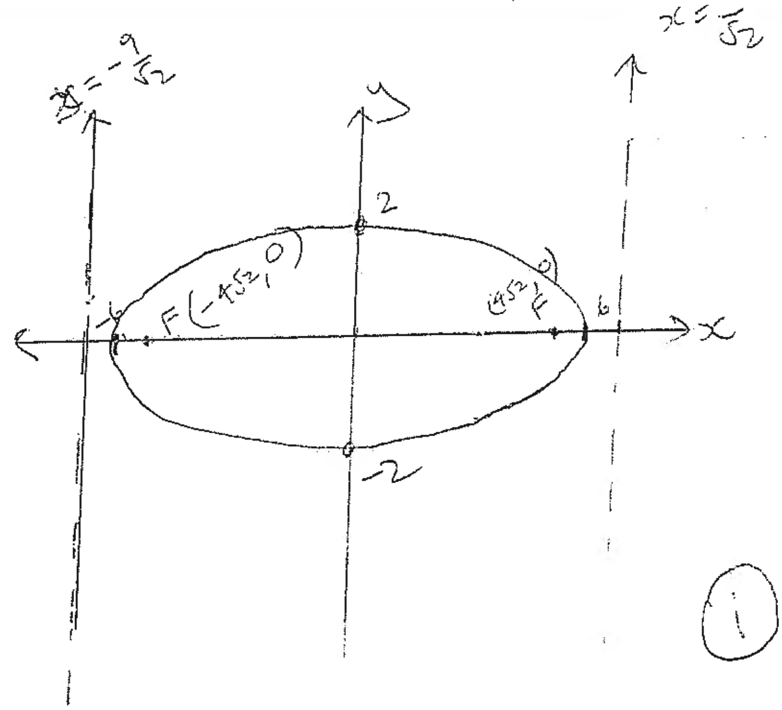
$e = \sqrt{1-\frac{4}{36}} = \sqrt{\frac{32}{36}}$

$\therefore e = \dfrac{2\sqrt{2}}{3}$ (1)

(ii) Foci: $(\pm ae, 0)$
$= (\pm 4\sqrt{2}, 0)$ (1)

(iii) Directrixes is $x = \pm\dfrac{a}{e} = \pm\dfrac{6}{2\sqrt{2}} \times 3 = \pm\dfrac{9\sqrt{2}}{2}$ (1)

(iv)



(1)

(e) (i) $\frac{x^2}{16} - \frac{y^2}{25} = 1$

$b^2 = a^2(e^2 - 1)$

$25 = 16(e^2 - 1)$

$e^2 = 1 + \frac{25}{16}$

$e = \pm \frac{\sqrt{41}}{4}$ but $e > 0$ for hyperbola

$\therefore e = \frac{\sqrt{41}}{4}$ only (1)

(ii) $y = \pm \frac{b}{a} x$ $\therefore y = \pm \frac{5}{4} x$ (1)

(iii) $|PS - PS'| = 2a$

$|2 - PS'| = 8$ (as $a = 4$)

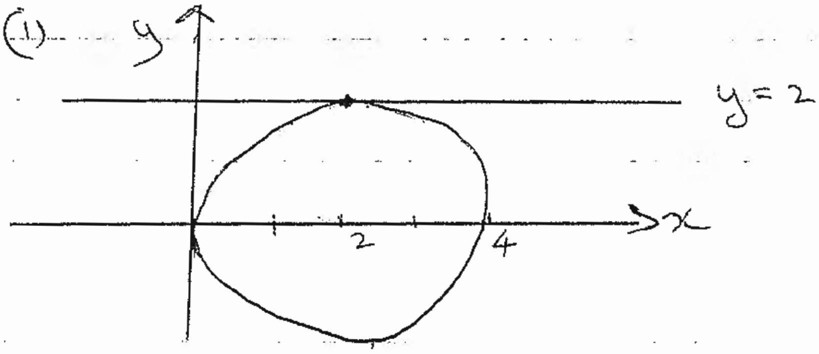
$PS' = 10$ or -6 but $PS' > 0$ (distance)

$\therefore PS' = 10$ only (1)

(a) $\frac{7-2i}{3+i} \times \frac{3-i}{3-i} = \frac{21-6i-7i-2}{9+1} = \frac{19}{10} - \frac{13i}{10}$ 1

Question 12

(b) (i)



1 line
1 circle.

(ii) line and circle intersect at $2+2i$ 1

(c) as $z\bar{z} = |z|^2$ $\therefore |z-w|^2 = (z-w)\overline{(z-w)}$

$= (z-w)(\bar{z}-\bar{w})$ 1

$= z\bar{z} - z\bar{w} - \bar{z}w + w\bar{w}$

$= 9 - (z\bar{w} + w\bar{z}) + 1$ 1

$= 10 - (z\bar{w} + w\bar{z})$

$=$ RHS.

(d)(i) $P(x) = 2x^3 + 3x^2 - x + 1$

roots are $\alpha^2, \beta^2, \gamma^2$

$y^2 = \alpha^2, \beta^2, \gamma^2$ $\quad y^2 = x$ where $x = \alpha^2, \beta^2, \gamma^2$

$\therefore y = \sqrt{x}$

$\therefore \boxed{Q(x) =} \ 2(\sqrt{x})^3 + 3(\sqrt{x})^2 - \sqrt{x} + 1 = 0$

$2(\sqrt{x})^3 - \sqrt{x} = -3x - 1$ 1

$\sqrt{x}(2x-1) = -(3x+1)$

$x(4x^2 - 4x + 1) = (3x+1)^2$

$4x^3 - 4x^2 + x = 9x^2 + 6x + 1$

$\therefore 4x^3 - 13x^2 - 5x - 1 = 0$

$\therefore Q(x) = 4x^3 - 13x^2 - 5x - 1$ 1 method 2 (α I α C)

$a+bi$ $\quad c+di$ $\quad \sqrt{a^2+b^2}=3$

$a^2+b^2=9$

$$LHS = |z-w|^2 = (a-c)^2+(b-d)^2 = a^2+c^2-2ac+b^2+d^2-2bd$$
$$= 9+1-2(ac+bd) \quad 1$$

$$RHS = 10-\left[(a+bi)(c-di)+(a-bi)(c+di)\right]$$
$$=10-\left[ac+bd+bci-adi+ac+bd-cbi+adi\right]$$
$$=10-\left[2ac+2bd\right]$$
$$=10-2(ac+bd) \quad 1$$
$$=LHS.$$

method 3

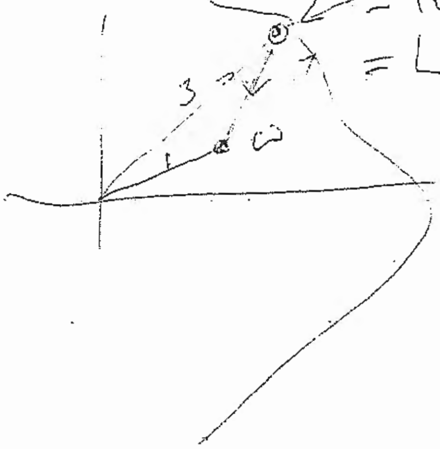
$3\cos\theta+3i\sin\theta$ $\quad \cos\alpha+i\sin\alpha$

let $z=3\,\text{cis}\,\theta$ $\quad w=1\,\text{cis}\,\alpha$

$$LHS = |z-w|^2 = (3\cos\theta-\cos\alpha)^2+(3\sin\theta-\sin\alpha)^2$$
$$=9\cos^2\theta+\cos^2\alpha-6\cos\theta\cos\alpha+9\sin^2\theta+\sin^2\alpha-6\sin\alpha\sin\theta$$
$$=9+1-6(\cos\theta\cos\alpha+\sin\theta\sin\alpha)$$
$$=10-6(\cos\theta\cos\alpha+\sin\theta.\sin\alpha)$$

$$RHS = 10-\left[(3\cos\theta+3i\sin\theta)(\cos\alpha-i\sin\alpha)+(3\cos\theta-3i\sin\theta)(\cos\alpha+i\sin\alpha)\right]$$
$$=10-\left[3\cos\theta\cos\alpha-i3\cos\theta\sin\alpha+3i\sin\theta\cos\alpha+3\sin\theta\sin\alpha+3\cos\theta\cos\alpha+3i\cos\theta\sin\alpha-3i\sin\theta\cos\alpha+3\sin\theta\sin\alpha\right]$$
$$=10-(6\cos\theta\cos\alpha+6\sin\theta\sin\alpha)$$
$$=10-6(\cos\theta\cos\alpha+\sin\theta.\sin\alpha) \quad 1$$
$$=LHS$$

method 4



(ii) $[P(\alpha) =]\ 2\alpha^3 + 3\alpha^2 - \alpha + 1 = 0$

$[P(\beta) =]\ 2\beta^3 + 3\beta^2 - \beta + 1 = 0$

$2\gamma^3 + 3\gamma^2 - \gamma + 1 = 0$

$$2(\alpha^3+\beta^3+\gamma^3) = -3(\alpha^2+\beta^2+\gamma^2) + (\alpha+\beta+\gamma) - 3$$

1

$\alpha^2+\beta^2+\gamma^2 = -b/a = \frac{13}{4}$ (from part (i))

$\alpha+\beta+\gamma = -3/2$

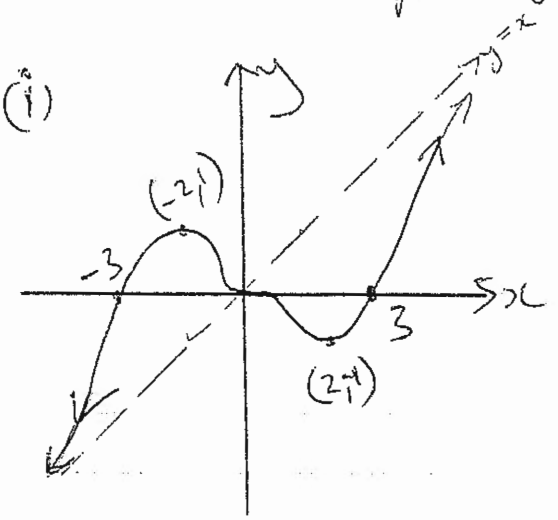
$$\therefore 2(\alpha^3+\beta^3+\gamma^3) = -3\times\frac{13}{4} + \frac{-3}{2} - 3$$

$$2(\alpha^3+\beta^3+\gamma^3) = -14\tfrac{1}{4}$$

$$\alpha^3+\beta^3+\gamma^3 = \frac{-57}{8}$$

1

(e) (i)



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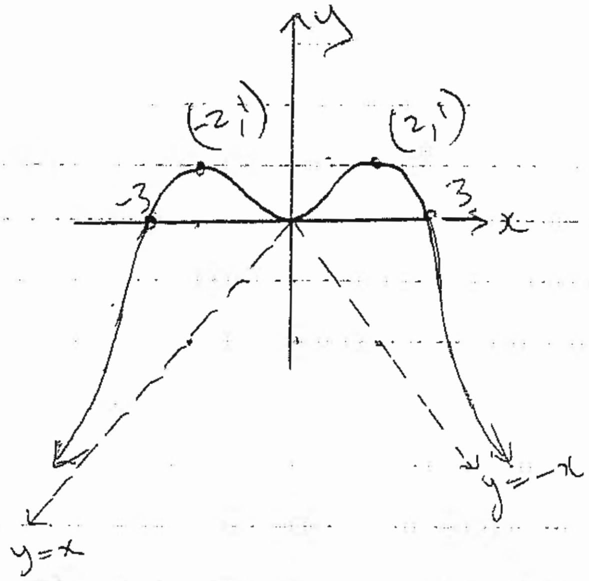
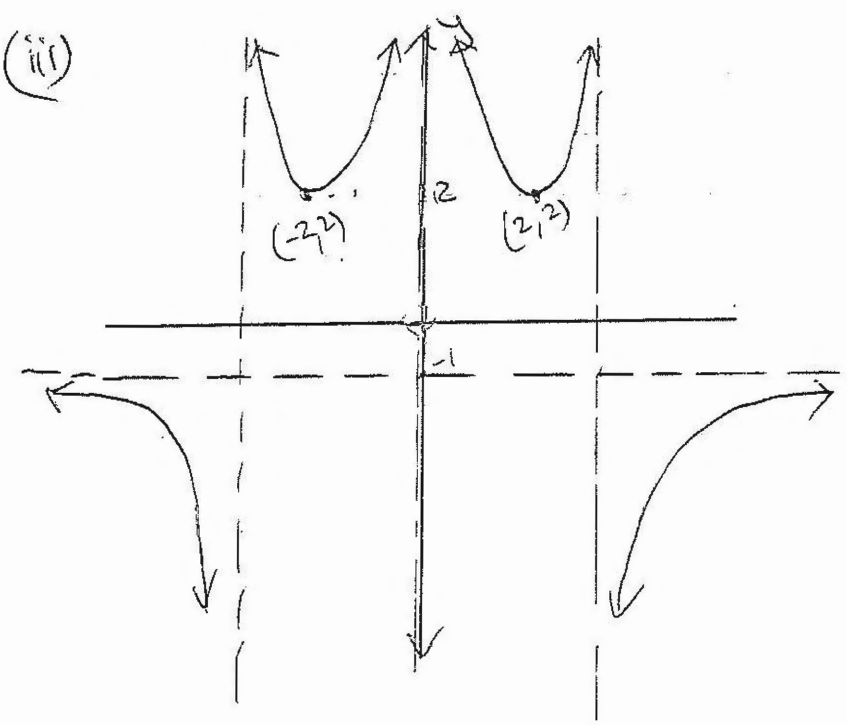
(ii)



1

(iii)



① for both branches above the x-axis

① For both branches below x-axis

① for $y=-1$; and turning pts $(\pm 2, 2)$ and a at $x=0$

Question 13

(a) $P(x) = x^4 - x^3 - 2x^2 + 6x - 4$

as $(1+i)$ is a zero so is $(1-i)$
(all coefficients are rational

$\therefore (x-(1+i))(x-(1-i))$ is a factor
ie $x^2 - 2x + 2$ is a factor ①

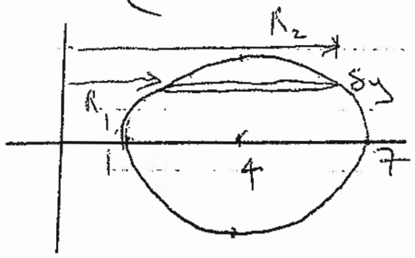
* now can either do a poly. division or use factor theorem.

$P(-2) = 16 + 8 - 8 - 12 - 4$
$= 0$
$\therefore (x+2)$ is a factor

$P(1) = 1 - 1 - 2 + 6 - 4$
$= 0$
$\therefore (x-1)$ is a factor

$\therefore P(x) = (x+2)(x-1)(x^2 - 2x + 2)$
$= (x+2)(x-1)(x-1-i)(x-1+i)$ ①

(b) (i)



$$\delta V = (\pi R_2^2 - \pi R_1^2)\delta y$$

$$x = 4 \pm \sqrt{1 - \frac{y^2}{4}}$$

$$R_2 + R_1 = 8 \qquad R_2 - R_1 = 6\sqrt{1 - \frac{y^2}{4}}$$

$$\therefore \delta V = \pi (R_2^2 - R_1^2)\,\delta y$$

$$= \pi (R_2 - R_1)(R_2 + R_1)\,\delta y \qquad (1)$$

$$= \pi \times 6\sqrt{1 - \frac{y^2}{4}} \times 8\ \delta y$$

$$= \pi\ \frac{48}{2}\sqrt{4 - y^2}\ \delta y \qquad (1)$$

$$= 8\pi\sqrt{36 - 9y^2}$$

(ii)

$$V = \lim_{\delta y \to 0} \sum_{-2}^{2} 8\pi\sqrt{36 - 9y^2}\ \delta y$$

$$= 8\pi \int_{-2}^{2} \sqrt{36 - 9y^2}\ dy$$

$$= 24\pi \int_{-2}^{2} \sqrt{4 - y^2}\ dy \qquad (1)$$

$$= 24\pi \times \frac{1}{2}(\pi \times 2^2)$$

(as $\sqrt{4-y^2}$ is a semi-circle)

$$= 48\pi^2 \text{ units}^3 \qquad (1)$$

(c)

$$18x^3 + 3x^2 - 28x + 12 = 0$$

let $P(x) = 18x^3 + 3x^2 - 28x + 12$

$$P'(x) = 56x^2 + 6x - 28$$

$$= 2(28x^2 + 3x - 14)$$

$$= 2(9x + 7)(3x - 2)$$

$P'(x) = 0$ when $x = -7/9$ or $2/3$ (1)

$$P(-7/9) = 18(-7/9)^3 + 3(-\tfrac{7}{9})^2 + 28(7/9) + 12 \neq 0$$

$P(2/3) = 18(2/3)^3 + 3 \times 4/9 - \frac{56}{3} + 12$

$= \frac{16}{3} + \frac{12}{9} - \frac{56}{3} + 12$

$= 0$

$\therefore (3x-2)$ is a double root (1)

ie $P(x) = (3x-2)^2(ax+\beta)$

$= (9x^2 - 12x + 4)(ax+\beta)$

by inspection $a=2$, $\beta = 3$

$\therefore P(x) = (3x-2)^2(2x+3)$

$\therefore$ roots are $2/3, 2/3, -3/2$ (1)

(d) (i) $P(5p, \frac{5}{p})$ $Q(5q, \frac{5}{q})$

$m = \frac{5/q - 5/p}{5q - 5p} = \frac{5\left(\frac{p-q}{pq}\right)}{5(q-p)} = \frac{-1}{pq}$ (1)

eqn of PQ is:

$y - 5/p = \frac{-1}{pq}(x - 5p)$

$pqy - 5q = -x + 5p$ (1)

$x + pqy = 5p + 5q$

$x + pqy = 5(p+q)$

(ii) length of PQ is K^2 (data)

$K^2 = \sqrt{5^2(p-q)^2 + 5^2\left(\frac{1}{p} - \frac{1}{q}\right)^2}$

$$m^4 = 5^2(p-q)^2 + 5^2\left(\frac{1}{p}-\frac{1}{q}\right)^2 \quad (1)$$

$$m^4 = 5^2(p-q)^2 + 5^2\frac{(-p+q)^2}{p^2q^2}$$

$$m^4 = 5^2(p-q)^2 + \frac{5^2}{p^2q^2}(p-q)^2(-1)^2$$

$$m^4 = 25(p-q)^2\left[1 + \frac{1}{p^2q^2}\right]$$

$$\left.\begin{aligned} p^2q^2m^4 &= 25(p-q)^2(p^2q^2+1) \\ p^2q^2m^4 &= 25(p^2+q^2-2pq)(p^2q^2+1) \end{aligned}\right\} \quad (1)$$

$$p^2q^2m^4 = 25\left[(p+q)^2 - 4pq\right](p^2q^2+1)$$

(iii) Midpoint of PQ is:

$$M = \left(\frac{5p+5q}{2}, \frac{5/p + 5/q}{2}\right)$$

$$\therefore R = \left(\frac{5(p+q)}{2}, \frac{5(p+q)}{2pq}\right) \qquad (R = M)$$

$$\therefore X = \frac{5}{2}(p+q) \qquad Y = \frac{5}{2}(p+q)\frac{1}{pq} \quad (1)$$

$$\therefore p+q = \frac{2x}{5} \quad \ldots (1) \qquad y = \frac{x}{pq}$$

$$pq = \frac{x}{y} \quad \ldots (2)$$

sub (1) and (2) into (ii)'s result.

$$\therefore \frac{x^2}{y^2}m^4 = 25\left[\frac{4x^2}{5^2} - \frac{4x}{y}\right]\left(\frac{x^2}{y^2}+1\right)$$

$$\frac{x^2}{y^2}m^4 = \left(4x^2 - \frac{4x.25}{y}\right)\left(\frac{x^2}{y^2}+1\right) \qquad \text{(now multiply both sides by } y^3\text{)}.$$

$$x^2ym^4 = (4x^2y - 4x.25)(x^2+y^2)$$

$$xym^4 = 4(xy-25)(x^2+y^2) \quad (1)$$

Question 14

(a) (i) $P(4\cos\theta, 3\sin\theta)$

$$\frac{dx}{d\theta} = -4\sin\theta \qquad \frac{dy}{d\theta} = 3\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \frac{3\cos\theta}{-4\sin\theta} \quad (1)$$

eqn of tangent at P is: $y - 3\sin\theta = \frac{-3\cos\theta}{4\sin\theta}(x - 4\cos\theta)$

$$y - 3\sin\theta = -\frac{3\cos\theta\, x}{4\sin\theta} + \frac{3\cos^2\theta}{\sin\theta}$$

$$\frac{3\cos\theta\, x}{4\sin\theta} + y = \frac{3\cos^2\theta}{\sin\theta} + 3\sin\theta$$

$$\frac{3\cos\theta\, x}{4\sin\theta} + y = \frac{3\cos^2\theta + 3\sin^2\theta}{\sin\theta}$$

$$\frac{3\cos\theta\, x}{4\sin\theta} + y = \frac{3}{\sin\theta}$$

$$(1) \quad \frac{\cos\theta\, x}{4} + \frac{y\sin\theta}{3} = 1 \ \ldots (1)$$

At Q; $(4\cos\theta, 4\sin\theta)$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \frac{4\cos\theta}{-4\sin\theta} = \frac{\cos\theta}{-\sin\theta} \quad (1)$$

eqn. of tangent at Q is: $y - 4\sin\theta = -\frac{\cos\theta}{\sin\theta}(x - 4\cos\theta)$

$$y + \frac{x\cos\theta}{\sin\theta} = \frac{4\cos^2\theta}{\sin\theta} + \frac{4\sin^2\theta}{\sin\theta}$$

$$y + x\frac{\cos\theta}{\sin\theta} = \frac{4}{\sin\theta}$$

$$(1) \quad \frac{\sin\theta}{4}y + \frac{x\cos\theta}{4} = 1 \ \ldots (2)$$

(ii) solve ① and ② simultaneously

$$y = \frac{3}{\sin\theta} - \frac{3\cos\theta x}{4\sin\theta} \quad \text{sub into ②}$$

$$\frac{3}{\sin\theta} - \frac{3\cos\theta x}{4\sin\theta} + \frac{x\cos\theta}{\sin\theta} = \frac{4}{\sin\theta}$$

$$3 - \tfrac{3}{4}\cos\theta x + x\cos\theta = 4$$

$$x\cos\theta\left(-\tfrac{3}{4}+1\right) = 1$$

$$x = \frac{4}{\cos\theta} \qquad ①$$

Sub. back

$$\therefore y = \frac{3}{\sin\theta} - \frac{3\cos\theta}{4\sin\theta}\cdot\frac{4}{\cos\theta}$$

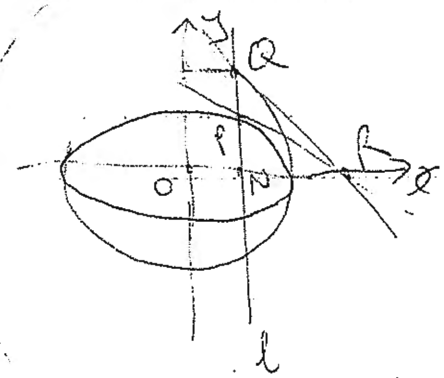
$$y = \frac{3}{\sin\theta} - \frac{3}{\sin\theta}$$

$$y = 0$$

$$\therefore R\left(\frac{4}{\cos\theta}, 0\right) \qquad ①$$

so R lies on the x-axis.

iii) ON . OR



$N(4\cos\theta, 0)$

$R\left(\frac{4}{\cos\theta}, 0\right)$

$$ON.OR = 4\cos\theta \times \frac{4}{\cos\theta}$$

① $ON.OR = 16$ which is independent of P and Q.

(b) (i) $\cot 3\theta = -1$

$\therefore \tan 3\theta = -1$

$3\theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}, \frac{19\pi}{4}, \frac{23\pi}{4}$

$\theta = \underbrace{\frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{15\pi}{12}}_{1}, \underbrace{\frac{19\pi}{12}, \frac{23\pi}{12}}_{1}$

(ii) $\cot 3\theta = -1$ and $\cot 3\theta = \dfrac{t^3 - 3t}{3t^2 - 1}$

$\therefore -1 = \dfrac{t^3 - 3t}{3t^2 - 1}$

$1 - 3t^2 = t^3 - 3t$

$0 = t^3 + 3t^2 - 3t - 1$

but $t = \cot\theta$ (data)

$\therefore$ solns for t are

$\cot \pi/4, \cot \frac{7\pi}{12}, \cot \frac{11\pi}{12}$ $\left(= \cot \frac{19\pi}{12}\right)$ $\left(= \cot \frac{23\pi}{12}\right)$

1 product of the roots is:

$\cot \pi/4 . \cot \frac{7\pi}{12} . \cot \frac{11\pi}{12} = -d/a = 1$

$\cot \pi/4 . \cot \frac{7\pi}{12} . \cot \frac{11\pi}{12} = 1 \quad \ldots (1)$

now $\cot(-\theta) = -\cot\theta$ (odd fn)

deduct 2π from each of the angles in (1)

$\therefore -\cot \pi/4 \times -\cot \frac{5\pi}{12} \times -\cot \frac{9\pi}{12} = 1$

1

$-1 \times \cot \pi/4 \times \cot \frac{5\pi}{12} \times \cot \frac{9\pi}{12} = 1$

$\therefore \cot \pi/4 . \cot \frac{5\pi}{12} . \cot \frac{9\pi}{12} = -1$

$\therefore \cot\left(\frac{\pi}{4}\right) . \cot\left(\frac{5\pi}{12}\right) \cot\left(\frac{9\pi}{12}\right)$

(iii) eqn with roots $\tan\frac{\pi}{12}$, $\tan\frac{5\pi}{12}$, $\tan\frac{9\pi}{12}$

ie $\frac{1}{\cot \pi/12}$, $\frac{1}{\cot\frac{5\pi}{12}}$, $\frac{1}{\cot\frac{9\pi}{12}}$

ie $\frac{1}{t}$ (1)

$\therefore$ eqn is: $\left(\frac{1}{t}\right)^3 + 3\left(\frac{1}{t}\right)^2 - 3\left(\frac{1}{t}\right) - 1 = 0$

$1 + 3t - 3t^2 - t^3 = 0$ (1)

$\therefore$ $\boxed{t^3 + 3t^2 - 3t - 1 = 0}$

(c) (i) $y = x^x$ for $x > 0$

$y = e^{x\ln x}$ $\Leftarrow$ $\left(\begin{array}{l}\ln y = \ln x^x \\ \ln y = x\ln x\end{array}\right)$

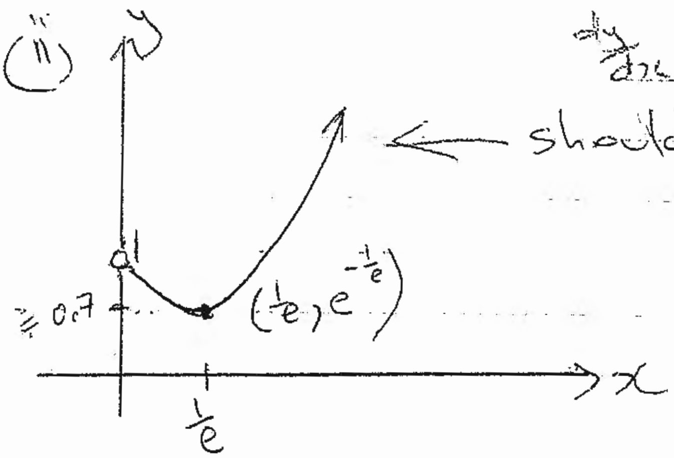
$\frac{dy}{dx} = e^{x\ln x} \times \left[x \times \frac{1}{x} + \ln x \times 1\right]$

$= e^{x\ln x}(1 + \ln x)$

$= e^{\ln x^x}(1+\ln x)$

$= x^x(1+\ln x)$ (1)

(ii)



$\frac{dy}{dx} = 0$ when $\ln x = -1$ $x = e^{-1}$

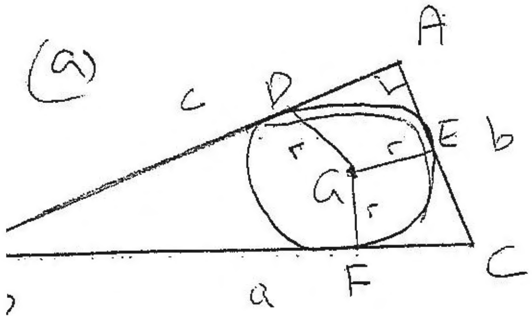
$\leftarrow$ should be quite steep

(1) for shape.

(1) for min. turning point and starting at $(0,1)$

Question 15

(a)



$\hat{ADG} = 90°$ (tangent meets radius at right angles)

Similarly $A\hat{E}G = 90°$ and $G\hat{F}C = 90°$

$\hat{A} = 90°$ (data)

$AD = AE$ (tangents from external point are equal.)

In ADGE,

all angles are $90°$ (proven above)

$AD = AE$ (proven above)

$\therefore$ ADGE is a square ①

so $AD = AE = DG = GE = r$

$\therefore$ $BD = c - r$

$BD = BF$ (tangents drawn from external point)

$\therefore$ $BF = c - r$. (as $BD = c - r$) ①

now $AE = r$

$\therefore$ $EC = b - r$

$CF = b - r$ (as $CF = EC$)

$\therefore$ $BC = BF + FC$ (sum of adjacent lengths)

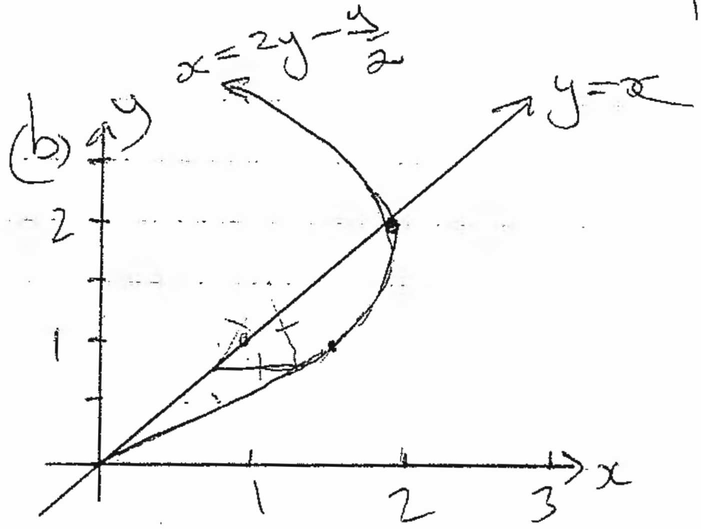
$a = c - r + b - r$ ①

$a = c + b - 2r$

$a - c - b = -2r$

$\therefore$ $r = \frac{1}{2}(b + c - a)$

(b)

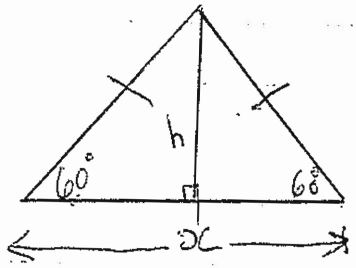


interepts:

$y=2y-\frac{y^2}{2}$

$y^2-2y=0$

$y=0$ or $y=2$



$h=\frac{\sqrt{3}}{2}x$

length of side of triangle $= x_2 - x_1$

$= 2y - \frac{y^2}{2} - y$

$= y - y^2/2$ (1)

Area of triangle $= \frac{1}{2}ab\sin 60°$

$= \frac{1}{2}x^2 \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}}{4}(y - y^2/2)^2$ (1)

Volume of single triangular prism:

$$\delta V = \frac{\sqrt{3}}{4}\left(y-\frac{y^2}{2}\right)^2 \delta y$$

$\therefore$ Volume of solid:

$$V = \lim_{\delta y \to 0} \sum_{y=0}^{2} \frac{\sqrt{3}}{4}(y - y^2/2)^2 \delta y$$

$$= \frac{\sqrt{3}}{4}\int_0^2 (y - y^2/2)^2 \, dy \quad (1)$$

$$= \frac{\sqrt{3}}{4}\int_0^2 \left(y^2 + \frac{y^4}{4} - y^3\right) dy$$

(ii) $= \frac{\sqrt{3}}{4}\left[\frac{1}{3}y^3 + \frac{y^5}{20} - \frac{1}{4}y^4\right]_0^2$ (1)

$= \frac{\sqrt{3}}{4}\left[\frac{1}{3}\times 8 + \frac{32}{20} - \frac{16}{4} - 0\right]$

$= \frac{\sqrt{3}}{4}\left[\frac{160}{60} + \frac{96}{60} - \frac{240}{60}\right] = \frac{\sqrt{3}}{4}\times\frac{16}{60}$

$= \frac{\sqrt{3}}{15}$ units^3 (1)

(c) (i)

$I_n = \int_0^1 (1-x^2)^{n/2}\,dx$

$= \left[x(1-x^2)^{n/2}\right]_0^1 + n\int_0^1 x^2(1-x^2)^{n/2-1}\,dx$ (1)

$= 0 - n\int_0^1 (1-x^2+1)(1-x^2)^{n/2-1}\,dx$

$= -n\int_0^1 (1-x^2)^{\frac{n}{2}}\,dx + \int_0^1 (1-x^2)^{n/2-1}\,dx$ (1)

$= -nI_n + nI_{n-2}$ (as $\frac{n}{2}-1 = \frac{n-2}{2}$) (1)

$I_n + nI_n = nI_{n-2}$

$I_n(1+n) = nI_{n-2}$

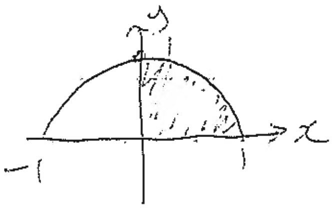
$I_n = \frac{n}{n+1}I_{n-2}$

(c) (ii) Evaluate I_5

$I_5 = \frac{5}{6} I_3$

$I_3 = \frac{3}{4} I_1$

$I_1 = \int_0^1 \sqrt{1-x^2}\, dx$



$\frac{1}{4}$ of a circle

$\therefore I_1 = \frac{1}{4}\pi \times 1^2 = \frac{\pi}{4}$ 1

$\therefore I_3 = \frac{3}{4} \times \frac{\pi}{4} = \frac{3\pi}{16}$

$I_5 = \frac{5}{6} \times \frac{3\pi}{16} = \frac{5\pi}{32}$ 1

(d) $(2-m^2) > 0$

$m^2 < 2$

$-\sqrt{2} < m < \sqrt{2}$ 1

but $m > 0$ given

$\therefore 0 < m < \sqrt{2}$ 1

Q16(a)(i) $a > 0$, $b > 0$ given $a \neq b$

now $(a-b)^2 \geq 0$

$a^2 + b^2 - 2ab \geq 0$ 1

$a^2 + b^2 \geq 2ab$

(ii) now $a^2 + b^2 \geq 2ab$

similarly $a^2 + c^2 \geq 2ac$

$b^2 + c^2 \geq 2bc$

$$\therefore 2(a^2+b^2+c^2) \geq 2(ab+ac+bc) \quad (1)$$
$$a^2+b^2+c^2 \geq ab+ac+bc$$

(iii) Let $a = \sin\alpha \quad b = \cos\alpha$

now $a^2+b^2 \geq 2ab$ from (i)
$$\sin^2\alpha + \cos^2\alpha \geq 2\sin\alpha\cos\alpha \quad (1)$$
$$\therefore \sin^2\alpha + \cos^2\alpha \geq \sin 2\alpha$$

(iv) $a^2+b^2+c^2 \geq ab+ac+bc$ from (ii)

let $a = \sin\alpha$, $b = \cos\alpha$, $c = \tan\alpha$

$$\therefore \sin^2\alpha + \cos^2\alpha + \tan^2\alpha \geq \sin\alpha\cos\alpha + \cos\alpha\tan\alpha + \tan\alpha\sin\alpha$$

$$(1) \quad \geq \frac{1}{2}\sin 2\alpha + \sin\alpha + \frac{\sin^2\alpha}{\cos\alpha}$$
$$= \frac{1}{2}\sin 2\alpha + \sin\alpha + \frac{1-\cos^2\alpha}{\cos\alpha}$$
$$= \frac{1}{2}\sin 2\alpha + \sin\alpha + \sec\alpha - \cos\alpha$$

(b) Paul has been chosen, 2 females

so committee is 1m, 2w

$$\text{Prob} = \frac{{}^{n-1}C_1}{{}^{n}C_2}$$
$$= \frac{(n-1)!}{(n-2)!} \times \frac{2!(n-2)!}{n!}$$
$$= \frac{2}{n} \quad (1)$$

(c) (i) $P(x) = (x^2 - a^2)Q(x) + px + q$

$= (x-a)(x+a)Q(x) + px + q$

$P(a) = pa + q$ --- (1)

$P(-a) = -ap + q$ -- (2)

(1) − (2)

$P(a) - P(-a) = pa + q + ap - q$
$= 2ap$ (1)

$\therefore p = \frac{1}{2a}[P(a) - P(-a)]$

(1) + (2)

$P(a) + P(-a) = pa + q - ap + q$
$= 2q$ (1)

$\therefore q = \frac{1}{2}[P(a) + P(-a)]$

ii) Given $P(x) = x^n - a^n$

if n is even; $P(a) = a^n - a^n = 0$
$P(-a) = (-a)^n - a^n = 0$

from (i) remainder is $\cancel{px} + q$
$p = 0,\ q = 0$
$\therefore$ no remainder (1)

if n is odd; $P(a) = a^n - a^n = 0$
$P(-a) = (-a)^n - a^n = -a^n - a^n = -2a^n$
$p = \frac{1}{2a}[0 - (-2a^n)] = a^{n-1}$
$q = \frac{1}{2}[0 + (-2a^n)] = -a^n$
so $R(x) = a^{n-1}x - a^n$ (1)

d) (i) $S_n = 1 + \sum_{r=1}^{n} \frac{1}{r!}$

$$e - S_n = e\int_0^1 \frac{x^n}{n!} e^{-x}\,dx$$

step one: Prove true for $n=1$

$LHS = e - S_1$
$= e - (1 + \frac{1}{1!})$
$= e - 2$

$RHS = e\int_0^1 \frac{x^1}{1!} e^{-x}\,dx$
$= e\int_0^1 x e^{-x}\,dx$
$= e\left[(-xe^{-x})\right]_0^1 + \int_0^1 e^{-x}\,dx$
$= e(-e^{-1} - 0) + \left[-e^{-x}\right]_0^1$
$= e(-e^{-1} - e^{-1} + e^0)$
$= e(-2e^{-1} + 1)$
$= e(-2\frac{1}{e} + 1)$
$= e - 2$
$= LHS.$ 1

step two: Assume true for $n=K$, K is an integer

ie $e - S_K = e\int_0^1 \frac{x^K}{K!} e^{-x}\,dx$

step three: Prove true for $n=K+1$,

ie $e - S_{K+1} = e\int_0^1 \frac{x^{K+1}}{(K+1)!} e^{-x}\,dx$

$LHS = e - S_{K+1}$

$= e - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{K!} + \frac{1}{(K+1)!}\right)$

$= e\int_0^1 \frac{x^K}{K!} e^{-x}\,dx - \frac{1}{(K+1)!}$ (by assumption) 1

$$RHS = e\int_0^1 \frac{x^{K+1}}{(K+1)!} e^{-x} dx$$

$u = x^{K+1}$ $\quad$ $v = -e^{-x}$

$\frac{du}{dx} = (K+1)x^K$ $\quad$ $\frac{dv}{dx} = e^{-x}$

$$= \frac{e}{(K+1)!}\left[-x^{K+1}e^{-x}\right]_0^1 - \frac{e}{(K+1)!}\int_0^1 (K+1)x^K e^{-x} dx$$

$$= \frac{e}{(K+1)!} \times -e^{-1} + \frac{e}{K!}\int_0^1 x^K e^{-x} dx \quad 1$$

$$= \frac{-1}{(K+1)!} + e\int_0^1 \frac{x^K}{K!} e^{-x} dx$$

$= LHS.$

Step Four: By the Principle of Mathematical Induction, the result is true for all integers, $n \geq 1$.

(ii) Deduce $0 < e - S_n < \frac{3}{(n+1)!}$

now $e\int_0^1 \frac{x^n}{n!} e^{-x} dx > 0$ as $e > 0$ and $e^{-x} > 0$

and $x^n \geq 0$ for $x > 0$

$e < 3$: $e\int_0^1 \frac{x^n}{n!} e^{-x} dx < 3\int_0^1 \frac{x^n}{n!} e^{-x} dx$

$< 3\int_0^1 \frac{x^n}{n!} dx$ as $e^{-x} \leq 1$

1 $\quad = 3\left[\frac{x^{n+1}}{n!(n+1)}\right]_0^1$ for $0 < x < 1$

$= \frac{3}{(n+1)!} - 0$

$\therefore 0 < e - S_n < \frac{3}{(n+1)!}$

(iii) Prove that $(e - S_n)n!$ isn't an integer

now $e - S_n < \frac{3}{n!(n+1)}$ from (ii)

$(e - S_n)n! < \frac{3}{n+1}$

$\frac{3}{n+1} \leq 1$ for $n = 2, 3, 4, \ldots$

$\therefore (e - S_n)n! < 1$ 1

$\therefore (e - S_n)n!$ is not an integer for $n = 2, 3, 4, \ldots$

(iv) Assume $e = p/q$

now $(e - S_n)n! = \left[e - \left(1 + \sum_{r=1}^{n} \frac{1}{r!}\right)\right]n!$

$= \left(\frac{p}{q} - 1 - \sum_{r=1}^{n} \frac{1}{r!}\right)n!$

$= \frac{pn!}{q} - n! - \left(\sum_{r=1}^{n} \frac{1}{r!}\right)n!$

now as q is any integer and $n! = n(n-1)(n-2) \times \ldots 3 \times 2$.

which includes "q", $\therefore n!$ is divisible by q

For $n > r$, then $n! \times \sum_{r=1} \frac{1}{r!}$ must be an integer

$\therefore \frac{pn!}{q} - n! - \left(\sum_{r=1} \frac{1}{r!}\right)n!$ is an integer 1

but this contradicts (iii) where $(e - S_n)n!$ is not an integer. $\therefore e \neq p/q$ so e is not rational.